

IITAKA'S $C_{n,m}$ CONJECTURE FOR 3-FOLDS IN POSITIVE CHARACTERISTIC

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ABSTRACT. In this paper, we prove that for a fibration $f : X \rightarrow Z$ from a smooth projective 3-fold to a smooth projective curve, over an algebraically closed field k with $\text{char } k = p > 5$, if the geometric generic fiber $X_{\overline{\eta}}$ is smooth, then subadditivity of Kodaira dimensions holds, i.e.

$$\kappa(X) \geq \kappa(X_{\overline{\eta}}) + \kappa(Z).$$

1. INTRODUCTION

Throughout this paper, a *fibration* means a projective morphism $f : X \rightarrow Y$ between varieties such that the natural morphism $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an isomorphism.

The Kodaira dimension is one of the most important birational invariants and plays a key role in the birational classification of algebraic varieties. For a fibration, we have the following conjecture on Kodaira dimensions, which was proposed by Iitaka in characteristic zero.

Conjecture 1.1 ($C_{n,m}$). *Let $f : X \rightarrow Z$ be a fibration between smooth projective varieties of dimension n and m respectively over an algebraically closed field k with $\text{char } k = p \geq 0$, whose geometric generic fiber $X_{\overline{\eta}}$ is integral and smooth. Then*

$$\kappa(X) \geq \kappa(X_{\overline{\eta}}) + \kappa(Z).$$

In characteristic zero, many results related to this conjecture are known [6, 11, 12, 14, 22, 23, 27, 28, 29, 31, 32, 34, 41, 42]. In particular, this conjecture was reduced to problems in the minimal model program by Kawamata [29, Corollary 1.2].

In positive characteristic, Conjecture 1.1 has been proved in some cases recently. Chen and Zhang showed $C_{n,n-1}$ [15, Theorem 1.2]. Under the assumption that $p > 5$, $C_{3,1}$ was shown when $k = \overline{\mathbb{F}_p}$ by Birkar, Chen and Zhang [9, Theorem 1.2], when $X_{\overline{\eta}}$ is of general type [19, Theorem 1.5] (see also [45, Appendix 7]), and when the genus of Z is at least two by Zhang [45, Corollary 1.9]. Furthermore, when f has singular geometric generic fiber, its dualizing sheaf, denoted by $\omega_{X_{\overline{\eta}}}$ (the same notation with canonical sheaf because they coincide with each other when $X_{\overline{\eta}}$ is smooth), was considered by [37] and [45], and under some special situations, an analogous inequality $\kappa(X) \geq \kappa(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}) + \kappa(Z)$ was proved.

The aim of this paper is to prove the theorem below.

Theorem 1.2. *Conjecture $C_{3,m}$ holds when $\text{char } k = p > 5$.*

The proof relies on the minimal model program for varieties of dimension at most three in characteristic $p > 5$ developed by several mathematicians including Birkar, Cascini, Hacon, Tanaka, Waldron and Xu. The case when $\kappa(X_{\overline{\eta}}) = 2$ has been

proved by the first author [19, Theorem 1.5]. In this paper we only need to consider the cases $\kappa(X_{\overline{\eta}}) = 0, 1$.

This paper is organized as follows. Section 2 includes some basic results to be used in our proof, including minimal model theory of 3-folds, vector bundles on elliptic curves and weak positivity of push-forward of pluri-relative canonical sheaves. Section 3 and 4 are devoted to study the cases $\kappa(X_{\overline{\eta}}) = 0$ and 1 respectively.

Notation and Conventions: In this paper, we fix an algebraically closed field k of characteristic $p > 0$. A k -scheme is a separated scheme of finite type over k . A *variety* means an integral k -scheme, and a *curve* (resp. *surface*, *n-fold*) means a variety of dimension one (resp. two, n).

Let $\varphi : S \rightarrow T$ be a morphism of schemes and let T' be a T -scheme. Then we denote by $S_{T'}$ and $\varphi_{T'} : S_{T'} \rightarrow T'$ respectively the fiber product $S \times_T T'$ and its second projection. For a prime $p \in \mathbb{Z}$, \mathbb{F}_p and $\mathbb{Z}_{(p)}$ denote respectively $\mathbb{Z}/p\mathbb{Z}$ and the localization of \mathbb{Z} at $p\mathbb{Z}$. For a Cartier, $\mathbb{Z}_{(p)}$ -Cartier or \mathbb{Q} -Cartier divisor D on S (resp. an \mathcal{O}_S -module \mathcal{G}), the pullback of D (resp. \mathcal{G}) to $S_{T'}$ is denoted by $D_{T'}$ or $D|_{S_{T'}}$ (resp. $\mathcal{G}_{T'}$ or $\mathcal{G}|_{S_{T'}}$) if it is well-defined. Similarly, for a homomorphism of \mathcal{O}_S -modules $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ the pullback of α to $S_{T'}$ is denoted by $\alpha_{T'} : \mathcal{F}_{T'} \rightarrow \mathcal{G}_{T'}$.

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2. PRELIMINARIES

In this section, we recall some basic results which will be used in the proof.

2.1. Minimal model of 3-fold. Existence of (log) minimal models of 3-fold in positive characteristic $p > 5$ was first proved for canonical singularities by Hacon and Xu [24], and in general by Birkar [7] (see [44] for the lc case). The result on Mori fiber spaces was proved for terminal singularities by Cascini, Tanaka and Xu [13], and in general by Birkar and Waldron [8]. We collect some results in the following theorem, which will be used in our proof.

Theorem 2.1. *Assume that the base field k has characteristic $p > 5$. Let $f : X \rightarrow Z$ be a contraction from a normal 3-fold, and let Δ be an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor on X .*

(1) If either (X, Δ) is klt and $K_X + \Delta$ is pseudo-effective over Z , or (X, Δ) is lc and $K_X + \Delta$ has a weak Zariski decomposition¹, then (X, Δ) has a log minimal model over Z .

(2) If (X, Δ) is a dlt pair and Z is a smooth projective curve with $g(Z) \geq 1$, then every step of LMMP in [7, Sec. 3.5-3.7] starting from (X, Δ) is over Z .

Proof. For (1) please refer to [7, Theorem 1.2 and Proposition 7.3].

For (2), since (X, Δ) is dlt, every $K_X + \Delta$ -extremal ray is generated by a rational curve by cone theorem [8, Theorem 1.1], which is contracted by f since $g(Z) \geq 1$. So for an extremal contraction $X \rightarrow \bar{X}$, if there is a divisorial contraction or a flip $\sigma : X \dashrightarrow X^+$ as in [7, Sec. 3.5-3.7], there exist natural morphism $\bar{f} : \bar{X} \rightarrow Z$ and $f^+ : X^+ \rightarrow Z$ fitting into the following commutative diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow f & \swarrow f^+ \\ & \bar{X} & \\ & \downarrow \bar{f} & \\ & Z & \end{array} .$$

Note that $(X^+, \Delta^+ = \sigma_*\Delta)$ is a dlt pair. We can show this assertion by induction. \square

2.2. Covering Theorem. The result below is [[26], Theorem 10.5] when X and Y are both smooth, and the proof there also applies when varieties are normal.

Theorem 2.2. ([26, Theorem 10.5]) *Let $f : X \rightarrow Y$ be a proper surjective morphism between complete normal varieties. If D is a Cartier divisor on Y and E an effective f -exceptional divisor on X , then*

$$\kappa(X, f^*D + E) = \kappa(Y, D).$$

As a corollary we get the following useful result.

Lemma 2.3. *Let $g : W \rightarrow Y$ be surjective projective morphism between projective varieties. Assume Y is normal and let $L_1, L_2 \in \text{Pic}^0(Y)$ be two line bundles on Y . If $g^*L_1 \sim_{\mathbb{Q}} g^*L_2$ then $L_1 \sim_{\mathbb{Q}} L_2$.*

Proof. Let $L = L_1 \otimes L_2^{-1}$. Denote by $\sigma : W' \rightarrow W$ the normalization and $g' = g \circ \sigma : W' \rightarrow Y$. Then $g'^*L \sim_{\mathbb{Q}} 0$. Applying Theorem 2.2 to $g' : W' \rightarrow Y$ gives that $L \sim_{\mathbb{Q}} 0$, which is equivalent to that $L_1 \sim_{\mathbb{Q}} L_2$. \square

2.3. Adjunction.

Lemma 2.4. *Assume that the base field k has characteristic $p > 5$. Let (X, Δ) be a normal, \mathbb{Q} -factorial, lc 3-fold (not necessarily projective). Let C be a projective lc center of (X, Δ) and \tilde{C} be the normalization of C . If $(K_X + \Delta)|_{\tilde{C}}$ is numerically trivial, then $(K_X + \Delta)|_{\tilde{C}}$ is \mathbb{Q} -trivial.*

¹i.e., there exists a birational projective morphism $\mu : W \rightarrow X$ such that $\nu^*(K_X + \Delta) = P + M$ where P is nef over Z and M is effective

Proof. By [7, Lemma 6.5], we can take a crepant partial resolution $\mu : X' \rightarrow X$ such that

$$K_{X'} + D + \Delta' \sim_{\mathbb{Q}} \mu^*(K_X + \Delta) \cdots (\clubsuit)$$

where D is a reduced irreducible divisor dominant over C and $(X', D + \Delta')$ is dlt. Then considering the restriction of the relation \clubsuit on D , by the adjunction formula [30, 5.3], we have

$$K_D + \Delta_D \sim_{\mathbb{Q}} \mu^*(K_X + \Delta)|_D.$$

Then D is a normal projective surface hence granted a natural morphism $D \rightarrow \tilde{C}$, and (D, Δ_D) is log canonical by [7, Lemma 4.2]. Applying [38, Theorem 1.2], we have that $K_D + \Delta_D$ is semi-ample, thus $\mu^*(K_X + \Delta)|_D$ is \mathbb{Q} -trivial since $(K_X + \Delta)|_{\tilde{C}}$ is numerically trivial. We can conclude that $(K_X + \Delta)|_{\tilde{C}}$ is \mathbb{Q} -trivial by Lemma 2.3. \square

2.4. Vector bundles on elliptic curves. In this subsection, we recall some facts about vector bundles on elliptic curves, which are used in the proof of Theorem 3.2.

Theorem 2.5. *Let C be an elliptic curve, and let $\mathcal{E}_C(r, d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and of degree d .*

- (1) ([2, Theorem 5]) *For each $r > 0$, there exists a unique element $\mathcal{E}_{r,0}$ of $\mathcal{E}_C(r, 0)$ with $H^0(C, \mathcal{E}_{r,0}) \neq 0$. Moreover, for every $\mathcal{E} \in \mathcal{E}_C(r, 0)$ there exists an $\mathcal{L} \in \text{Pic}^0(C)$ such that $\mathcal{E} \cong \mathcal{E}_{r,0} \otimes \mathcal{L}$.*
- (2) ([35, Corollary 2.9]) *When the Hasse invariant $\text{Hasse}(C)$ is nonzero, $F_C^* \mathcal{E}_{r,0} \cong \mathcal{E}_{r,0}$. When $\text{Hasse}(C)$ is zero, $F_C^* \mathcal{E}_{r,0} \cong \bigoplus_{1 \leq i \leq \min\{r,p\}} \mathcal{E}_{[(r-i)/p]+1,0}$, where $[r]$ denotes the round down of r .*

Theorem 2.6 ([33, 1.4. Satz]). *Let \mathcal{E} be a vector bundle on a smooth projective curve C . If $F_C^e \mathcal{E} \cong \mathcal{E}$ for some $e > 0$, then there exists an étale morphism $\pi : C' \rightarrow C$ from a smooth projective curve C' such that $\pi^* \mathcal{E} \cong \bigoplus \mathcal{O}_{C'}$.*

Proposition 2.7. *Let \mathcal{E} be a vector bundle on an elliptic curve C . Then there exists a finite morphism $\pi : C' \rightarrow C$ from an elliptic curve C' such that $\pi^* \mathcal{E}$ is a direct sum of line bundles.*

Proof. We may assume that for every finite morphism $\varphi : B \rightarrow C$ from an elliptic curve B , $\varphi^* \mathcal{E}$ is indecomposable. Set $d := \deg \mathcal{E}$ and $r := \text{rank} \mathcal{E}$. We show that $r = 1$. Let $Q \in C$ be a closed point. Replacing \mathcal{E} by $((r_C)^* \mathcal{E})(-dQ)$, we may assume that $d = 0$. Here $r_C : C \rightarrow C$ is the morphism given by multiplication by r . Hence Theorems 2.5 and 2.6 imply that when the Hasse invariant of C is nonzero (resp. zero), there exists an étale morphism $\pi : C' \rightarrow C$ (resp. an $e > 0$) such that $\pi^* \mathcal{E}$ (resp. $F_C^e \mathcal{E}$) is a direct sum of line bundles. This implies that $r = 1$. \square

2.5. Weak positivity. The following positivity result will be used in the proof of the case when the geometric generic fiber has Kodaira dimension one.

Theorem 2.8. *Assume that $\text{char} = p > 5$. Let $f : X \rightarrow Z$ be a fibration from a smooth projective 3-fold to a smooth projective curve. Suppose that the geometric generic fiber $X_{\overline{\eta}}$ has at most rational double points as singularities. If $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}}) = 1$, then there exists a real number $c > 0$ such that $f_* \omega_{X/Z}^m$ contains a nef subbundle of rank at least cm for sufficiently divisible $m > 0$.*

Before proving Theorem 2.8, we recall some results.

Theorem 2.9 ([15, 3.2]). *Let $f : X \rightarrow Z$ be a surjective morphism between smooth projective varieties, over an algebraically closed field of positive characteristic, whose geometric generic fiber is a smooth elliptic curve. Then $\kappa(X, K_{X/Z}) \geq 0$.*

The following lemma will be frequently used.

Lemma 2.10 ([43, Lemma 3.2]). *Let $f : X \rightarrow Z$ be a fibration between normal quasi-projective varieties. Let L be a f -nef \mathbb{Q} -Cartier divisor on X such that $L_\eta \sim_{\mathbb{Q}} 0$ where η is the generic point of Z . Assume $\dim Z \leq 3$. Then there exist a diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{\psi} & Z \end{array}$$

with ϕ, ψ projective birational, and an \mathbb{Q} -Cartier divisor D on Z' such that $\phi^*L \sim_{\mathbb{Q}} f'^*D$. Furthermore, if f is flat and Z is \mathbb{Q} -factorial, then we can take $X' = X$ and $Z' = Z$.

The next lemma is a consequence of Tanaka's vanishing theorem for surfaces [39].

Lemma 2.11. *Let $g : Y \rightarrow Z$ be a generically smooth surjective morphism from a smooth projective surface to a smooth projective curve. Let H be a nef and g -big divisor on Y . Then $g_*\mathcal{O}_Y(K_{Y/Z} + lH)$ is a nef vector bundle for every $l \gg 0$.*

Proof. Let A be an ample divisor on Z with $\deg A \geq \deg K_Z + 2$. Then $A - K_Z - z$ is ample for a closed point $z \in Z$ where z is seen as a divisor on Z . Note that $\nu(H) \geq 1$ and $H + g^*(A - K_Z - z)$ is nef and big. Denote by Y_z the fiber of g over z . By [39, Theorem 2.6] we see that

$$H^1(Y, K_Y + H + g^*(A - K_Z) + (l-1)H - Y_z) = H^1(Y, K_Y + H + g^*(A - K_Z - z) + (l-1)H) = 0$$

for $l \gg 0$. Thus for a closed point $z \in Z$, by the long exact sequence arising from taking cohomology of the exact sequence below

$$0 \rightarrow \mathcal{O}_Y(K_{Y/Z} + g^*A + lH - Y_z) \rightarrow \mathcal{O}_Y(K_{Y/Z} + g^*A + lH) \rightarrow \mathcal{O}_Y(K_{Y/Z} + g^*A + lH)|_{Y_z} \rightarrow 0$$

we conclude that the restriction

$$H^0(Y, K_{Y/Z} + g^*A + lH) \rightarrow H^0(Y_z, (K_{Y/Z} + g^*A + lH)|_{Y_z})$$

is surjective. This implies that $(g_*\mathcal{O}_Y(K_{Y/Z} + lH))(A)$ is generically globally generated. On the other hand, if z is general then Y_z is smooth, applying [36, Corollary 2.23], since $H|_{Y_z}$ is ample, we have that for $l \gg 0$ the morphism

$$H^0(Y_z, \phi_{Y_z}^{(e)} \otimes \mathcal{O}_{Y_z}(K_{Y_z} + lH_z)) : H^0(Y_z, K_{Y_z} + lp^e H_z) \rightarrow H^0(Y_z, K_{Y_z} + lH_z)$$

is surjective. This implies that the homomorphism ([19, Section 2])

$$gz_* (\phi_{Y/Z}^{(e)} \otimes \mathcal{O}_{Y_{Z^e}}((K_{Y/Z} + lH)_{Z^e})) \otimes \mathcal{O}_{Z^e}(A) :$$

$$g_*\mathcal{O}_Y(K_{Y/Z} + lp^e H + g^*(A + z)) \cong g_*\mathcal{O}_Y(K_{Y/Z} + lp^e H) \otimes \mathcal{O}_{Z^e}(A)$$

$$\rightarrow gz_* \mathcal{O}_{Y_{Z^e}}((K_{Y/Z} + lH)_{Z^e}) \otimes \mathcal{O}_{Z^e}(A + z) \cong F_Z^{e*} g_*\mathcal{O}_Y(K_{Y/Z} + lH) \otimes \mathcal{O}_{Z^e}(A)$$

is generically surjective. Thus for every $e > 0$, $F_Z^{e*}(g_*\mathcal{O}_Y(K_{Y/Z} + lH)) \otimes \mathcal{O}_{Z^e}(A)$ is generically globally generated, and hence is nef. We conclude that $g_*\mathcal{O}_Y(K_Y + lH)$ is nef by applying [19, Proposition 4.7]. \square

Proof of Theorem 2.8. Let W be a minimal model of X over Z . Let $\rho : X_{\bar{\eta}} \rightarrow W_{\bar{\eta}}$ be the induced morphism. Since $\rho_*\mathcal{O}_{X_{\bar{\eta}}} \cong \mathcal{O}_{W_{\bar{\eta}}}$, $W_{\bar{\eta}}$ is normal. Furthermore, since W is terminal, we have $K_{X_{\bar{\eta}}} \geq \rho^*K_{W_{\bar{\eta}}}$, and hence $W_{\bar{\eta}}$ has at most canonical singularities. In particular, replacing X with a minimal model, with the loss of smoothness we may assume that $K_{X/Z}$ is f -nef.

Then by [38, Theorem 1.2], $K_{X_{\bar{\eta}}}$ is semi-ample, and since $p > 5$, the geometric generic fiber of the Iitaka fibration $I_{\bar{\eta}} : X_{\bar{\eta}} \rightarrow C_{\bar{\eta}}$ is a smooth elliptic curve over $k(\bar{\eta})$ by [4, Theorem 7.18]. For the generic fiber X_{η} and sufficiently divisible positive integer n , since $H^0(X_{\bar{\eta}}, nK_{X_{\bar{\eta}}}) \cong H^0(X_{\eta}, nK_{X_{\eta}}) \otimes_{k(\eta)} k(\bar{\eta})$, we see that the Iitaka fibration $I_{\bar{\eta}} : X_{\bar{\eta}} \rightarrow C_{\bar{\eta}}$ coincides with the Iitaka fibration $I_{\eta} : X_{\eta} \rightarrow C_{\eta}$ tensoring with $k(\bar{\eta})$. Thus the geometric generic fiber of I_{η} is a smooth elliptic curve.

Considering the relative Iitaka fibration of $f : X \rightarrow Z$, whose geometric generic fiber is a smooth elliptic curve, we get a birational morphism $u : X' \rightarrow X$, a fibration $g : Y \rightarrow Z$ with Y smooth, and an elliptic fibration $h : X' \rightarrow Y$ fitting into the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ h \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z. \end{array}$$

Note that the geometric generic fiber $C_{\bar{\eta}}$ of $g : Y \rightarrow Z$ is normal, and hence smooth. By Lemma 2.10, we may assume that $u^*K_{X/Z} \sim_{\mathbb{Q}} h^*H$ for a nef g -big \mathbb{Q} -Cartier divisor on Y . By Theorem 2.9, we have $\kappa(X', K_{X'/Y}) \geq 0$, and hence there exists an injective homomorphism $h^*\omega_{Y/Z}^m \rightarrow \omega_{X'/Z}^m$ for sufficiently divisible $m > 0$. Let $l \gg 0$ be an integer such that lH is Cartier and $u^*lK_{X/Z} \sim h^*lH$. Then we have natural homomorphisms

$$\begin{aligned} (g_*\mathcal{O}_Y(K_{Y/Z} + lH))^{\otimes m} &\rightarrow g_*\mathcal{O}_Y(m(K_{Y/Z} + lH)) \cong g_*h_*\mathcal{O}_{X'}(mh^*(K_{Y/Z} + lH)) \\ &\hookrightarrow f_*u_*\mathcal{O}_{X'}(mK_{X'/Z} + u^*lmK_{X/Z}) \cong f_*\mathcal{O}_X(m(l+1)K_{X/Z}). \end{aligned}$$

Replacing l if necessary, we may assume that the first homomorphism is generically surjective. By Lemma 2.11, $g_*\mathcal{O}_Y(K_{Y/Z} + lH)$ is nef, and hence so is $g_*\mathcal{O}_Y(m(K_{Y/Z} + lH))$. This completes the proof. \square

3. THE CASE $\kappa(X_{\bar{\eta}}) = 0$

In this section, we prove Theorem 1.2 in the case when the Kodaira dimension of the geometric generic fiber is equal to zero. It is proved as a consequence of Theorems 3.1 and 3.2.

Theorem 3.1 ([20]). *Let $f : X \rightarrow Z$ be a surjective morphism between normal projective varieties over an algebraically closed field of characteristic $p > 0$, and let Δ be an effective \mathbb{Q} -divisor on X such that $a\Delta$ is integral for some $a > 0$ not divisible by p . Assume that $X_{\bar{\eta}}$ is Gorenstein and $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure, where $\bar{\eta}$ is the geometric generic point of Z . If $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + L)$ for some \mathbb{Q} -Cartier divisor L on Z , then L is pseudo-effective.*

Theorem 3.1 follows from [20, Theorem 4.2] (by setting $D = -(K_Z + L)$), and it is also proved by Patakfalvi [36, Theorem 1.6] when Z is a curve.

Theorem 3.2. *With the same notation and assumptions as in Theorem 3.1, if Z is an elliptic curve, then L is semi-ample.*

Proof. By Theorem 3.1, we have $\deg L \geq 0$. We may assume that $\deg L = 0$, and it suffices to show that $L \sim_{\mathbb{Q}} 0$. Since $(K_X + \Delta)_{\bar{\eta}} \sim_{\mathbb{Q}} 0$, there is an ample Cartier divisor A on X such that $l(K_X + \Delta)_{\bar{\eta}} + A_{\bar{\eta}}$ is ample and free for every $l \in a\mathbb{Z}$. Recall that $0 < a \in \mathbb{Z} \setminus p\mathbb{Z}$ and $a\Delta$ is integral. By Fujita's vanishing theorem, there exist some $m_0 > 0$ such that for every nef Cartier divisor N on $X_{\bar{\eta}}$, $\mathcal{O}_{X_{\bar{\eta}}}((m_0 - 1)A_{\bar{\eta}} + N)$ is 0-regular with respect to $l(K_X + \Delta)_{\bar{\eta}} + A_{\bar{\eta}}$ for every $l \in a\mathbb{Z}$. Then the natural homomorphism

$$\begin{aligned} H^0(X_{\bar{\eta}}, l(K_X + \Delta)_{\bar{\eta}} + mA_{\bar{\eta}}) \otimes H^0(X_{\bar{\eta}}, (m' - 1)A_{\bar{\eta}}) &\otimes H^0(X_{\bar{\eta}}, l'(K_X + \Delta)_{\bar{\eta}} + A_{\bar{\eta}}) \\ &\rightarrow H^0(X_{\bar{\eta}}, (l + l')(K_X + \Delta)_{\bar{\eta}} + (m + m')A_{\bar{\eta}}) \end{aligned}$$

is surjective for every $l, l' \in a\mathbb{Z}$ and $m, m' \geq m_0$. Thus

$$\begin{aligned} H^0(X_{\bar{\eta}}, l(K_X + \Delta)_{\bar{\eta}} + mA_{\bar{\eta}}) \otimes H^0(X_{\bar{\eta}}, l'(K_X + \Delta)_{\bar{\eta}} + m'A_{\bar{\eta}}) \\ \rightarrow H^0(X_{\bar{\eta}}, (l + l')(K_X + \Delta)_{\bar{\eta}} + (m + m')A_{\bar{\eta}}) \end{aligned}$$

is also surjective, and hence the natural homomorphism

$$\mathcal{G}(l, m) \otimes \mathcal{G}(l', m') \rightarrow \mathcal{G}(l + l', m + m')$$

is generically surjective, where $\mathcal{G}(l, m) := f_*\mathcal{O}_X(l(K_{X/Z} + \Delta) + mA)$. From now on, we use the same notation as [19, Sections 2 or 3] or [20, Section 2]. Replacing m_0 if necessary, by [36, Corollary 2.23] we may assume that

$$\begin{aligned} H^0(X_{\bar{\eta}}, \phi_{(X_{\bar{\eta}}, \Delta_{\bar{\eta}})}^{(e)} \otimes \mathcal{O}_{X_{\bar{\eta}}}(N + m_0A_{\bar{\eta}})) : \\ H^0(X_{\bar{\eta}}, (1 - p^e)(K_X + \Delta)_{\bar{\eta}} + p^e(N + m_0A_{\bar{\eta}})) \rightarrow H^0(X_{\bar{\eta}}, N + m_0A_{\bar{\eta}}) \end{aligned}$$

is surjective for every $e > 0$ with $a|(p^e - 1)$ and for every nef Cartier divisor N on $X_{\bar{\eta}}$. Since $l(K_X + \Delta)_{\bar{\eta}}$ is nef,

$$\begin{aligned} f_{Z^e*}(\phi_{(X/Z, \Delta)}^{(e)} \otimes \mathcal{O}_{X_{Z^e}}(l(K_{X/Z} + \Delta)_{Z^e} + m_0A_{Z^e})) : \\ \mathcal{G}((l - 1)p^e + 1, m_0p^e) \rightarrow f_{Z^e*}\mathcal{O}_{X_{Z^e}}(l(K_{X/Z} + \Delta)_{Z^e} + m_0A_{Z^e}) \cong F_Z^{e*}\mathcal{G}(l, m_0) \end{aligned}$$

is generically surjective. Let $b > 0$ be an integer such that $a|b$, that bL is integral and that $b(K_X + \Delta)$ is linearly equivalent to bf^*L . By Proposition 2.7, there exists a finite morphism $\pi : Z' \rightarrow Z$ from an elliptic curve Z' such that $\pi^*\mathcal{G}(r, m_0)$ is a

direct sum of line bundles for each $0 \leq r < b$ with $a|r$. By Lemma 2.3, we may replace L and $\mathcal{G}(r, m_0)$ respectively by its pullback by π . Set

$$\begin{aligned}\mathcal{F} &:= \bigoplus_{0 \leq r < b, a|r} \mathcal{G}(r, m_0), \\ \mu &:= \min\{\deg \mathcal{M} | \mathcal{M} \in \text{Pic}(Z) \text{ and } \mathcal{M} \text{ is a direct summand of } \mathcal{F}\}, \text{ and} \\ T &:= \{\mathcal{M} \in \text{Pic}(Z) | \deg \mathcal{M} = \mu \text{ and } \mathcal{M} \text{ is a direct summand of } \mathcal{F}\} \\ &= \{\mathcal{M}_1, \dots, \mathcal{M}_\lambda\}.\end{aligned}$$

Then for every $\mathcal{M}_i \in T$, there exists an $0 \leq s < b$ with $a|s$ such that the composition

$$\begin{aligned}\mathcal{G}(s, m_0)^{\otimes p^e-1} \otimes \mathcal{G}(r_{i,e}, m_0) \otimes \mathcal{O}_Z(-q_{i,e}bL) \\ \rightarrow \mathcal{G}((s-1)p^e + 1, p^e m_0) \rightarrow F_Z^{e*} \mathcal{G}(s, m_0) \twoheadrightarrow \mathcal{M}_i^{p^e}\end{aligned}$$

is generically surjective for every $e > 0$ with $a|(p^e - 1)$. Here $q_{i,e}$ and $r_{i,e}$ are integers satisfying $1 + s - p^e = -q_{i,e}b + r_{i,e}$ and $0 \leq r_{i,e} < b$. Then there exist a line bundle \mathcal{M} which is a direct summand of $\mathcal{G}(s, m_0)^{p^e-1} \otimes \mathcal{G}(r_{i,e}, m_0)$ and a non-zero morphism $\mathcal{M} \rightarrow \mathcal{M}_i^{p^e}(q_{i,e}bL)$. By considering the degree of the line bundles, we see that $\mathcal{M}_i^{p^e}(q_{i,e}bL) \cong \mathcal{M} \in T^{p^e}$, where

$$T^n := \{\bigotimes_{1 \leq i \leq \lambda} \mathcal{M}_i^{n_i} \in \text{Pic}(Z) | n_i \geq 0, \sum_{1 \leq i \leq \lambda} n_i = n\}.$$

Fix an integer $e > 0$ such that $a|p^e - 1$. Set $n := \lambda(p^e - 1) + 1$. For every $\mathcal{N} \in T^n$, there exist $n_1, \dots, n_\lambda \geq 0$ such that $\mathcal{N} \cong \bigotimes_{1 \leq i \leq \lambda} \mathcal{M}_i^{n_i}$ and $n'_j := n_j - p^e \geq 0$ for at least one j . Then

$$\mathcal{N}(q_{j,e}bL) \cong \left(\bigotimes_{i \neq j} \mathcal{M}_i^{n_i}\right) \otimes \mathcal{M}_j^{n'_j} \otimes \mathcal{M}_j^{p^e}(q_{j,e}bL).$$

Since $\mathcal{M}_j^{p^e}(q_{j,e}bL) \in T^{p^e}$, we have $\mathcal{N}(q_{j,e}bL) \in T^n$. Hence for every $m \geq q := \max\{q_{1,e}, \dots, q_{\lambda,e}\}$,

$$\mathcal{N}(mbL) \in \{\mathcal{M}(kbL) \in \text{Pic}(Z) | \mathcal{M} \in T^n, 0 \leq k < q\}.$$

Since T^n is a finite set, there are integers $m > m' > 0$ such that $\mathcal{N}(mbL) \cong \mathcal{N}(m'bL)$, and hence $(m - m')bL \sim 0$. \square

Proof of Theorem 1.2: the case $\kappa(X_{\overline{\eta}}) = 0$. As in the proof of Theorem 2.8, we may assume that X is minimal over Z and $K_{X_{\overline{\eta}}}$ is semi-ample, thus $K_{X_{\overline{\eta}}} \sim_{\mathbb{Q}} 0$. By Lemma 2.10, K_X is \mathbb{Q} -linearly equivalent to the pullback of $K_Z + L$ for some \mathbb{Q} -divisor L on Z . In particular $\kappa(X, K_X) = \kappa(Z, K_Z + L)$. It is enough to show that $\kappa(Z, K_Z + L) \geq \kappa(Z)$. By Lemma 3.1, we see that L is nef. Note that since $X_{\overline{\eta}}$ has at most rational double points as singularities, $X_{\overline{\eta}}$ is Gorenstein and $p > 5$, $X_{\overline{\eta}}$ is F -pure by [3, Section 3] and [21]. When Z is of general type, by Theorem 3.1, we have $K_Z + L$ is big, thus $\kappa(Z, K_Z + L) = \dim Z = \kappa(Z)$. And when Z is an elliptic curve, by Theorem 3.2, we have $\kappa(Z, K_Z + L) \geq \kappa(Z)$. This completes the proof. \square

4. THE CASE $\kappa(X_{\bar{\eta}}) = 1$

In this section, we consider the case when the Kodaira dimension of the geometric generic fiber is one.

Proof of Theorem 1.2: the case $\kappa(X_{\bar{\eta}}) = 1$. Let $f : X \rightarrow Z$ be a surjective morphism from a smooth projective 3-fold to a smooth projective curve of genus at least one, and let $\bar{\eta}$ be the geometric generic point of Z . Suppose that $\kappa(X_{\bar{\eta}}) = 1$. With the loss of smoothness, we assume that X is a minimal model. Then $X_{\bar{\eta}}$ has canonical singularities by the proof of Theorem 2.8.

If $g(Z) > 1$, then since $f_*\omega_{X/Z}^m$ contains a nef sub-bundle of rank $\geq cm$ for some $c > 0$ and any sufficiently divisible m (Theorem 2.8), by some standard arguments (proof of [9, Proposition 5.1]), we can conclude that

$$\kappa(X) \geq 2 = \kappa(Z) + \kappa(X_{\bar{\eta}}).$$

So from now on, we assume $g(Z) = 1$. Then $\omega_X = \omega_{X/Z}$. We break the proof into several steps.

Step 1: By considering the relative Iitaka fibration and applying Lemma 2.10, we get the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X \\ h \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

where Y is a smooth projective surface, and h is fibration with geometric fiber being a smooth elliptic curve by the proof of Theorem 2.8, such that $\sigma^*K_X \sim_{\mathbb{Q}} h^*D$ where D is a nef g -big divisor on Y .

If D is big, then we are done. From now on, we assume the numerical dimension $\nu(D) = 1$. Then we claim that

Claim. If X has a fibration $f' : X \rightarrow W$ to a normal projective curve W such that $K_{F'}$ is numerically trivial, where F' denotes the generic fiber of f' . Assume moreover that there exist $L \in \text{Pic}^0(Z)$ and an integer $m > 0$ such that $h^0(X, mK_X + f^*L) > 0$. Then K_X is semi-ample.

Proof of the claim. Take an effective divisor $D_L \sim mK_X + f^*L$. Since D_L is nef, effective and $D_L|_{F'} \sim_{\text{num}} 0$, we have

$$(mK_X + f^*L)|_{F'} \sim D_L|_{F'} \sim 0.$$

By Lemma 2.10 we can assume $D_L \sim_{\mathbb{Q}} f'^*A$ where A is a divisor on W , which is ample since $D_L > 0$. So we only need to show that $L \sim_{\mathbb{Q}} 0$.

Since X has at most terminal singularities, it is smooth in codimension one, so F' is a regular surface over the function field $K(W)$ of W . Applying [40, Theorem 0.2], since $K_{F'}$ is numerically trivial, we have $K_{F'} \sim_{\mathbb{Q}} 0$. Therefore, we conclude that

$$f^*L|_{F'} \sim_{\mathbb{Q}} mK_{F'} + f^*L|_{F'} \sim_{\mathbb{Q}} (mK_X + f^*L)|_{F'} \sim_{\mathbb{Q}} D_L|_{F'} \sim_{\mathbb{Q}} 0.$$

On the other hand, F' is dominant over the curve $Z \otimes_k K(W)$, passing to the algebraic closure of $K(W)$ and applying Lemma 2.3, we show that L is torsion. \square

Step 2: By Theorem 2.8, there exists $c > 0$ such that for sufficiently divisible m_1 , $f_*\omega_X^{m_1}$ contains a nef sub-bundle V of rank $r_{m_1} \geq cm_1$. If $\deg V > 0$, then we are done by some standard arguments ([9, Propostion 5.1]). So we assume that $\deg V = 0$, thus by Proposition 2.7 there exists a flat base change between two elliptic curves $\pi : Z_1 \rightarrow Z$ such that $\pi^*V = \oplus_{i=1}^n \mathcal{L}_i$ where $\mathcal{L}_i \in \text{Pic}^0(Z_1)$. Let X_1 be the normalization of $X \times_Z Z_1$. Then we get the following commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\pi_1} & X \\ f_1 \downarrow & & \downarrow f \\ Z_1 & \xrightarrow{\pi} & Z \end{array}$$

where π_1 and f_1 denote the natural projections. We have that $\pi^*f_*\omega_X^{m_1} \subset f_{1*}\pi_1^*\omega_X^{m_1}$ by [25, Proposition 9.3], thus

$$\pi^*V = \oplus_{i=1}^n \mathcal{L}_i \subset f_{1*}\pi_1^*\omega_X^{m_1}.$$

So we conclude that

$$h^0(X_1, \pi_1^*m_1K_X - f_1^*\mathcal{L}_i) \geq 1,$$

and if $\mathcal{L}_i = \mathcal{L}_j$ for some $j \neq i$ then the strict inequality holds. Since $\pi^* : \text{Pic}^0(Z) \rightarrow \text{Pic}^0(Z_1)$ is surjective, there exists L'_i such that $\mathcal{L}_i \sim \pi^*L'_i$, thus

$$\pi_1^*m_1K_X - f_1^*\mathcal{L}_i \sim \pi_1^*(m_1K_X + f^*L'_i).$$

Applying Theorem 2.2, we can find a sufficiently divisible integer $l > 0$ such that

$$h^0(X, l(m_1K_X - f^*L'_i)) \geq 1.$$

Put $m = lm_1$ and $L_i = lL'_i$. Then $h^0(X, mK_X - f^*L_i) \geq 1$.

If $h^0(X, mK_X - f^*L_i) > 1$, then $h^0(Y, mD - g^*L_i) > 1$ by the construction in Step 1. Since $mD - g^*L_i$ is nef and $\nu(mD - g^*L_i) = 1$, the movable part of $|mD - g^*L_i|$ has no base point, hence induces a fibration $g' : Y \rightarrow W'$ on Y to a curve W' . The Stein factorization of the composite morphism $g' \circ h : X' \rightarrow W'$ induces a fibration $f'' : X' \rightarrow W$ from X' to a normal curve W , which is defined by the base point free linear system $|\mu^*l(mK_X - f^*L_i)|$ for sufficiently divisible integer $l > 0$. Since $\sigma : X' \rightarrow X$ is a birational morphism such that $\sigma_*\mathcal{O}_{X'} = \mathcal{O}_X$, we conclude that $|\mu^*l(mK_X - f^*L_i)| = \mu^*|l(mK_X - f^*L_i)|$, thus $|l(mK_X - f^*L_i)|$ has no base point, hence defines such a fibration $f' : X \rightarrow W$ as in Claim of Step 1. So K_X is semi-ample, and this completes the proof in this case.

From now on, we can assume $h^0(X, mK_X - f^*L_i) = 1$ and $h^0(X_1, \pi_1^*(mK_X - f^*L_i)) = 1$. For every i , we have unique effective divisors $B_i \sim mK_X - f^*L_i$. And by construction, we can assume $\pi_1^*B_i \neq \pi_1^*B_j$ if $i \neq j$, thus $L_i \neq L_j$. In the following, we only need to show that at least two of L_i are torsion.

Step 3: For every j , we have unique effective divisor $B_j \sim mK_X - f^*L_j$. Let B' be the reduced divisor supported on the union of components of $\sum_j B_j$. Take a smooth log resolution $\mu : \tilde{X} \rightarrow X$ of the pair (X, B') . Denote by $\tilde{f} : \tilde{X} \rightarrow Z$ the natural morphism. Let \tilde{B} be the reduced divisor supported on the union components of

$\sum_j \mu^* B_j$. Consider the dlt pair (\tilde{X}, \tilde{B}) . Since X has terminal singularities, there exists an effective μ -exceptional divisor E on \tilde{X} such that

$$K_{\tilde{X}} + \tilde{B} = \mu^* K_X + E + \tilde{B}$$

which is a weak Zariski decomposition of $K_{\tilde{X}} + \tilde{B}$. By Theorem 2.1, (\tilde{X}, \tilde{B}) has a minimal model (\hat{X}, \hat{B}) which is dlt, and there exists a natural morphism $\hat{f}: \hat{X} \rightarrow Z$. By the construction, we have the following:

(1) Note that $B_j|_{X_{\bar{\eta}}}$ is contained in finitely many fibers of the Iitaka fibration $I_{\bar{\eta}}: X_{\bar{\eta}} \rightarrow C_{\bar{\eta}}$, which implies that $\kappa(\tilde{X}_{\bar{\eta}}, (K_{\tilde{X}} + \tilde{B})|_{X_{\bar{\eta}}}) = 1$. Since the restriction $(K_{\hat{X}} + \hat{B})|_{\hat{X}_{\bar{\eta}}}$ is semi-ample by [38, Theorem 1.2], it induces an elliptic fibration on $\hat{X}_{\bar{\eta}}$ by construction. So applying Lemma 2.10 again, we get the following commutative diagram

$$\begin{array}{ccc} \hat{X}' & \xrightarrow{\hat{\sigma}} & \hat{X} \\ \hat{h} \downarrow & & \downarrow \hat{f} \\ \hat{Y} & \xrightarrow{\hat{g}} & Z \end{array}$$

where \hat{Y} is a smooth projective surface and \hat{h} is an elliptic fibration such that, $\hat{\sigma}^*(K_{\hat{X}} + \hat{B}) \sim_{\mathbb{Q}} \hat{h}^* \hat{D}$ where \hat{D} is a nef and \hat{g} -big divisor on \hat{Y} .

(2) We claim that $\nu(K_{\hat{X}} + \hat{B}) = \nu(\hat{D}) = 1$. Indeed, otherwise, \hat{D} will be big, and applying Theorem 2.2 we get a contradiction as follows

$$\begin{aligned} 2 &= \kappa(\hat{Y}, \hat{D}) = \kappa(\hat{X}', \hat{\sigma}^*(K_{\hat{X}} + \hat{B})) = \kappa(\hat{X}, K_{\hat{X}} + \hat{B}) = \kappa(\tilde{X}, K_{\tilde{X}} + \tilde{B}) \\ &\leq \kappa(\tilde{X}, K_{\tilde{X}} + \mu^* nm K_X - \sum_j L_j) \cdots \text{since } \tilde{B} \leq \mu^* \sum_j B_j \sim \mu^* nm K_X - \sum_j L_j \\ &\leq \kappa(\tilde{X}, \mu^* K_X + \tilde{E} + \mu^* nm K_X - \sum_j L_j) \\ &\quad \cdots \text{by } K_{\tilde{X}} \leq \mu^* K_X + \tilde{E} \text{ for some effective } \mu\text{-exceptional divisor } E \\ &= \kappa(X, \mu^*(nm + 1)K_X - \sum_j L_j) \leq \nu(X, (nm + 1)K_X - \sum_j L_j) = 1. \end{aligned}$$

(3) For sufficiently divisible M and every $0 \leq i \leq n$, by Step 1 and 2, there exists unique effective Cartier divisor

$$\hat{B}_i \sim M(mK_{\hat{X}} - \hat{f}^* L_i) + Mm\hat{B} \sim Mm(K_{\hat{X}} + \hat{B}) - M\hat{f}^* L_i.$$

(4) Take effective divisors $\hat{D}_i \sim Mm\hat{D} - M\hat{g}^* L_i$. Since \hat{D} is nef and $\hat{D}^2 = 0$, by [5, VIII.3] we can write that

$$\hat{D}_i = a_{i1}\hat{D}'_1 + \cdots + a_{ik}\hat{D}'_k, a_{i*} \in \mathbb{Z}^{\geq 0}$$

where the \hat{D}'_i 's are nef connected, primitive Cartier divisors such that $\hat{D}'_i{}^2 = 0$ and $\hat{D}'_i \cap \hat{D}'_j = \emptyset$ for $i \neq j$. Note that at least one of \hat{D}'_i is dominant over Z , hence intersects every fiber of \hat{f} , so by [5, VIII.4] we conclude that every \hat{D}'_i is dominant over Z .

(5) Take two divisors \hat{D}_1, \hat{D}_2 . Since $\hat{D}_1 \sim_{num} \hat{D}_2$ but $\hat{D}_1 \neq \hat{D}_2$, we have $\hat{D}_1 + \hat{D}_2$ contains at least two disjoint fundamental cycles. It is easy to see that we may assume $a_{11} > a_{21} \geq 0$ and $a_{22} > a_{12} \geq 0$. Let

$$\hat{G}_1 = \hat{\sigma}_*(\hat{h}^*\hat{D}'_1) \text{ and } \hat{G}_2 = \hat{\sigma}_*(\hat{h}^*\hat{D}'_2).$$

By construction, we have $\hat{\sigma}^*\hat{B}_i = \hat{h}^*\hat{D}_i$ and $\hat{D}'_1 \cap \hat{D}'_2 = \emptyset$, and in turn we conclude that $\hat{G}_1 \cap \hat{G}_2 = \emptyset$. We can write that

$$\hat{B}_1 = a_{11}\hat{G}_1 + a_{12}\hat{G}_2 + \hat{G}'_3 \text{ and } \hat{B}_2 = a_{21}\hat{G}_1 + a_{22}\hat{G}_2 + \hat{G}''_3$$

where neither of \hat{G}_1 and \hat{G}_2 intersects $\hat{G}'_3 \cup \hat{G}''_3$.

Step 4: By construction, we can find lc centers \hat{C}_1, \hat{C}_2 of (\hat{X}, \hat{B}) dominant over Z , contained in \hat{G}_1, \hat{G}_2 respectively. Denote by \hat{C}'_i the normalization of \hat{C}_i . Then by Lemma 2.4, we have $(K_{\hat{X}} + \hat{B})|_{\hat{C}'_i} \sim_{\mathbb{Q}} 0$. Therefore,

$$\begin{aligned} (1) \quad & -a_{21}M\hat{f}^*L_1|_{\hat{C}'_1} \sim_{\mathbb{Q}} a_{21}(Mm(K_{\hat{X}} + \hat{B}) - M\hat{f}^*L_1)|_{\hat{C}'_1} \\ & \sim_{\mathbb{Q}} a_{21}\hat{B}_1|_{\hat{C}'_1} \sim_{\mathbb{Q}} a_{11}a_{21}\hat{G}_1|_{\hat{C}'_1} \\ & \sim_{\mathbb{Q}} a_{11}\hat{B}_2|_{\hat{C}'_1} \sim_{\mathbb{Q}} -a_{11}M\hat{f}^*L_2|_{\hat{C}'_1} \end{aligned}$$

which, by Lemma 2.3, implies that

$$a_{21}ML_1 \sim_{\mathbb{Q}} a_{11}ML_2.$$

In the same way, restricting on \hat{C}'_2 gives

$$a_{22}ML_1 \sim_{\mathbb{Q}} a_{12}ML_2.$$

Finally by conditions $a_{11} > a_{21}$ and $a_{12} < a_{22}$, we conclude that $L_1 \sim_{\mathbb{Q}} L_2 \sim_{\mathbb{Q}} 0$, and this completes the proof. \square

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